Contents

1 Relations **5**
1.1 Binary relations . . . 5
1.2 Equivalence relations . . . 9
1.3 Operations on Relations . . . 13
1.3.1 Set operations . . . 14
1.3.2 Transposition . . . 14
1.3.3 Composition . . . 15
1.3.4 Closures . . . 19
1.4 Exercises . . . 21

2 Graphs **27**
2.1 Directed Graphs . . . 27
2.2 Undirected Graphs . . . 28
2.3 A more compact notation for undirected graphs . . . 30
2.4 Additional notions and some properties . . . 30
2.5 Connectivity . . . 32
2.5.1 Paths . . . 32
2.5.2 Path concatenation . . . 34
2.5.3 The triangular inequality . . . 35
2.5.4 Connected components . . . 36
2.6 Cycles . . . 37
2.6.1 Directed cycles . . . 37
2.6.2 Undirected cycles . . . 38
2.7 Euler and Hamilton cycles . . . 39
2.7.1 Euler cycles . . . 39
2.7.2 Hamilton cycles . . . 41
2.7.3 A theorem on Hamilton cycles . . . 41
2.7.4 A proof by contradiction . . . 42
2.7.5 A more explicit proof . . . 42
2.7.6 Proof of the Core Property . . . 43
2.8 Ramsey's theorem . . . 46
2.8.1 Introduction . . . 46
2.8.2 Ramsey's theorem . . . 47
2.8.3 A few applications . . . 50
2.9 Trees . . . 51
2.9.1 Undirected trees . . . 51
2.9.2 Rooted trees . . . 53
2.10 Exercises . . . 55

3 Functions **58**
3.1 Functions . . . 58
3.2 Equality of functions . . . 59
3.3 Monotonicity of function types . . . 59

3.4 Function composition 60
3.5 Lifting a function 61
3.6 Surjective, injective, and bijective functions 63
3.7 Inverse functions 65
3.8 Finite sets and counting 67
3.9 Exercises 71

4 Posets and lattices **73**
4.1 Partial orders 73
4.2 Extreme elements 76
4.3 Upper and lower bounds 79
4.4 Lattices 82
4.4.1 Definition 82
4.4.2 Algebraic properties 84
4.4.3 Distributive lattices 85
4.4.4 Complete lattices 86
4.5 Exercises 88

5 Monoids and Groups **91**
5.1 Operators and their properties 91
5.2 Semigroups and monoids 93
5.3 Groups 95
5.4 Subgroups 99
5.5 Cosets and Lagrange's Theorem 100
5.6 Permutation Groups 102
5.6.1 Function restriction and extension 102
5.6.2 Continued Compositions 103
5.6.3 Bijections 103
5.6.4 Permutations 104
5.6.5 Swaps 105
5.6.6 Neighbor swaps 107
5.7 Exercises 109

6 Combinatorics: the Art of Counting **111**
6.1 Introduction 111
6.2 Recurrence Relations 118
6.2.1 An example 118
6.2.2 The characteristic equation 119
6.2.3 Linear recurrence relations 121
6.2.4 Summary 123
6.3 Binomial Coefficients 124
6.3.1 Factorials 124
6.3.2 Binomial coefficients 125
6.3.3 The Shepherd's Principle 127
6.3.4 Newton's binomial formula 128

6.4 A few examples . . . 129
6.4.1 Summary . . . 131
6.5 Exercises . . . 131

7 Number Theory **135**
7.1 Introduction . . . 135
7.2 Divisibility . . . 135
7.3 Greatest common divisors . . . 138
7.4 Euclid's algorithm and its extension . . . 142
7.5 The prime numbers . . . 145
7.6 Modular Arithmetic . . . 149
7.6.1 Congruence relations . . . 149
7.6.2 An application: the nine and eleven tests . . . 152
7.7 Fermat's little theorem . . . 153
7.8 Cryptography: the RSA algorithm . . . 154
7.9 Exercises . . . 156